The path to spherical trigonometry

The sun does never set nor rise. When people think the sun is setting (it is not so). For having arrived at the end of the day, it makes itself produce two opposite effects, making night for what is below and day for what is on the other side.

Aitareya Brāhmaṇa 3.44 (Haug, 1863, p.242)

The oldest known records that discuss astronomical terms are the Vedas, records of hymns in ancient Indian literature. The Vedas were first transferred orally and were recorded later. The oldest, *Ṛgveda*, followed by *Yajurveda*, probably has its source in the period circa 1500 BCE, however it could even be based on older sources.

THE PATH TO SPHERICAL TRIGONOMETRY

Gnomon

The invention of the *gnomon*, a shadow casting rod, is attributed a similar role for the development of natural sciences as the invention of the wheel on an axle for the development of mechanics in general.[1] It is not known where the gnomon, one of the oldest astronomical devices or tools, was first used for astronomical purposes, but we know that it was used very early on in Egypt, Mesopotamia, China and India. In approximately the 9th or 10th millennium BCE, the summer monsoon from central Africa moved north and provided a modest but sufficiently large amount of rainfall in the southwestern Egyptian desert, which allowed for the survival of the nomadic tribes and their livestock. In the late Neolithic period, in Nabta, the landscape was covered with monsoon lakes, about a hundred kilometres west of Abu Simbel, a sort of ritual centre, and nomadic tribes used a stone megalith, perhaps as a gnomon, to mark direction, which assisted them in migrations.[2] Nabta was inhabited until the 4th millennium BCE, when it started to transform into a desert again as a result of climate change. One of the oldest known Egyptian records on the measurement of the length of a shadow originates from the period of the Middle Egyptian kingdom and speaks of the inability to measure the length of a shadow in the case of a solar eclipse.[3]

In Ancient India, during the civilization in the Indus valley, the directions of the sky may have been determined with the naked eye by observing the sunrise and the setting of the stars or constellations. From the oldest surviving written sources of *Vedas* and *Brahmanas*, the use of astronomical instruments for astronomical observations is not apparent.[4] The oldest known Indian record of the use of the gnomon, named *śaṅku* or *śaṅku-yantra* or *nara-yantra* in Sanskrit,

1 Couprie, 2011, Heaven and Earth in Ancient Greek Cosmology: From Thales to Heraclides Ponticus, p.41.

2 Wendorf, Schild, 2001, Holocene Settlement of the Egyptian Sahara: Volume 1: The Archaeology of Nabta Playa, pp.489–502. Malville, McKim et al., 2008, Astronomy of Nabta Playa. African Cultural Astronomy, Holbrook et al. (eds.), pp.39–51.

3 Daressy, 1901, Catalogue général des antiquités égyptiennes du Musée du Caire 25001–25385. Ostraca. See Borchard, 1920, Altägyptische Zeitmessung, pp.27 and 64–65, and text and footnote 1. The Middle Kingdom dates from the 11th dynasty when Nebhepetre Mentuhotep II brought together Lower and Upper Egypt, up to the 13th dynasty, about the middle of the 21st century BCE to the middle of the 17th century BCE.

4 Ôhashi, 1993, Development of Astronomical Observation in Vedic and Post-Vedic India, pp.193–197. *Ṛgveda*, *Sāmaveda*, *Yajurveda*, *Atharvaveda* and *Brahmanas*, the prose comments on the *Vedas* do not make notes on astronomical instruments. The text also mentions some false interpretations of mentions of gnomons from older texts. For an additional explanation, see also Holbrook and Baleisis, 2008, Naked-eye Astronomy for Cultural Astronomers. African Cultural Astronomy, Holbrook et al. (eds.), pp.53–75.

is a description of determining the directions of the sky in the *Kātyāyana Śulba Sūtra*, written around 200 BCE.[5] In view of the numerous geometric patterns on older archaeological finds, the gnomon was probably known in India well before then. From the *Vedāṅga* texts, which are an addition to the *Vedas* and describe mathematical and astronomical concepts, the use of water clocks in astronomical observations is also evident.[6] Astronomical concepts are addressed by *Jyotiṣa Vedāṅga*,[7] and the shadow of the gnomon (*pauruṣī*) is also mentioned in other old Indian texts such as *Candra-prajnapti* and *Uttarādhyayana*.[8]

The best sources of Mesopotamian astronomical heritage are records of various astronomical texts, such as ENŪMA ANU ENLIL[9] from the period around 1200 BCE and MUL.APIN.[10] In the MUL.APIN collection of clay tablets from the 7th century BCE, notes on the length of the shadow of a gnomon with the length of one *cubit* are also preserved. Despite some uncertainties, there is no doubt that the values are valid for the vertical gnomon with a length of one cubit or about 0.5 m, throwing a shadow on a horizontal surface.[11] The gnomon shadow length chart applies fairly well for locations between 35° and 36° N, which corresponds to the position of the Assur site, from which the results of the astronomical observations originally

5 Ôhashi, 1993, pp.206–207. The text of *Kātyāyana Śulba Sūtra* is supposed to have been written by Kātyāyana, about whom we know practically nothing, around 200 BCE but most likely he was not the (only) author. In literature, references from *Aitareya Brāhmaṇa*, XVIII.4 are mentioned as the oldest example of the gnomon in India, which however cannot be confirmed, Ôhashi, 1993, pp.204–205, also note 12, p.240. *Kātyāyana Śulba Sūtra* was written much later than the similar, but mathematically more significant texts of the *Āpastamba Śulba Sūtra* (6th century BCE) and *Baudhāyana Śulba Sūtra* (8th century BCE), which is also the oldest well-known mathematical text from India.

6 The *Vedāṅga* texts describe six branches of knowledge: phonetics, *Śikṣā*, grammar, *Vyākoraṅa*, etymology, *Nirukta*, metrics, *Chandas*, astronomy, *Jyotiṣa*, and rules of rituals and celebrations, *Kalpa*, divided into four parts - public celebrations, *Śrauta Sūtras*, celebrations, *Gṛyhya Sūtras*, law, *Dharma Sūtras*, and instructions for making altars, *Śulba Sūtras*. *Jyotiṣa Vedāṅga* is the main source of astronomical data from the Vedic period, while the *Śulba Sūtras* describe Vedic geometry. *Jyotiṣa Vedāṅga* may have been created around the year 1400 BCE. The exact time of the creation is unknown and the origins are probably somewhere in the north of India, between 27° and 29° N; see, Ôhashi, 1993, p.202.

7 For more information on the gnomon in India, see Subbarayappa, Sarma, 1985, Indian Astronomy. A Source Book, Chapter 15, pp.182–195, Sen, Shukla (eds.), 1985, History of Astronomy in India, pp.346–351, and Ôhashi, 1994, Astronomical Instruments and Classical Siddhāntas, pp.168–196. For water clocks, see Mishra, 2005, Vedanga Jyotisham, pp.104–107 and 170–173.

8 *Candra-prajnapti* probably originates from the period between the 2nd and 4th centuries BCE. Also, *Uttarādhyayana*, the first of the texts of *Mūla Sūtras*, XXVI, 13–14, Ôhashi, 1993, p.202.

9 ENŪMA ANU ENLIL is a collection of 70 (or 68, depending on the interpretation) tables with numerous astronomical notes. The source of the oldest records dates back to the beginning of the first ancient Babylonian dynasty (beginning of the 2nd millennium BCE). The translation of the title ENŪMA ANU ENLIL reads: *"In the time of Anu and Enlil ..."* The Sumerian god *An*, Akadic *Anu*, was the father of the gods and the god of heaven, the heavens. *Enlil* also belongs at the very top of the Mesopotamian deities; he was the god of the atmosphere and the winds. For a more detailed explanation of Table XIV, see Al Rawi, George, 1991/1992, Enūma Anu Enlil XIV and Other Early Astronomical Tables, Archiv für Orientforschung, Bd. 38/39, pp.52–73.

10 The name MUL.APIN consists of the initial words on the tables. It means the constellation of the Triangle and the star Andromeda δ. Weidner, 1924, Ein Babylonisches Compendium Der Himmelskunde, The American Journal of Semitic Languages and Literatures, vol. 40, no. 3, pp.186–208. For a detailed explanation, see Hunger, Pingree, 1989, Mul.Apin. An Astronomical Compendium in Cuneiform. Archiv für Orientforschung Beiheft 24 in Watson, Horrowitz, 2011, Writing Science before the Greeks: A Naturalistic Analysis of the Babylonian Astronomical Treatise MUL.APIN. The data, which was probably written on tablets around 700 BCE, is an older source.

11 The cubit, lat. *cubitum*, an ancient measure of length, corresponds to approximately 0.5 m. The lengths of the ancient cubits differ from one another. For the Mesopotamian cubit, see Reallexikon der Vorgeschichte/8, 1927, p.60, or Unger, 1916, Zwei Babylonische Antiken aus Nippur, Publikationen der Kaiserlich Osmanischen Museen 1. For the lengths of the shadow, see van der Waerden, 1968, Anfänge der Astronomie, p.80.

written by Assyrian astronomers and later recorded on tablets are supposed to have originated.[12] According to the sources, the gnomon from Mesopotamia, where it was certainly used by the 15th century BCE, is supposed to have been brought to Greece, along with the sundial, by ANAXIMANDER. HERODOTUS states that the Greeks were acquainted from the Babylonians with the *polos* (gr. Πόλος), *gnomon* (gr. Γνώμων) and twelve parts of the day,[13] and SUIDAS[14] stated that ANAXIMANDER is thought to have brought this knowledge to Greece. From DIOGENES LAERTIUS' work, which mentions FAVORINUS,[15] it is evident that ANAXIMANDER used a shadow catcher (gr. σκιοθήρων) in Sparta, which is understood to be a synonym for the gnomon.[16] ANAXIMANDER is supposed to have used the gnomon primarily to observe equinoxes and solstices. The gnomon was also certainly known to THALES, as he was supposed to have measured the height of the pyramid in Egypt, which he could only have done with a gnomon, though it is not clear whether THALES became acquainted with the gnomon in Egypt or elsewhere.

The sexagesimal system

The first written source of the existence of a measurement system in Mesopotamia, originally intended to measure distances and afterwards to measure angles or arcs, is Table XIV of the ENŪMA ANU ENLIL collection. We do not know the Sumerian measurement system in its entirety; however, the significance of their measurement quantities and units is fairly well understood from archaeological fragments.[17] Probably by the early 3rd millennium BCE, the Sumerians were using different basic measurement units, including the unit for distance, *danna*, where distance was measured indirectly as the time needed to travel the measured distance.[18] The

12 Van der Waerden, 1949, Babylonian Astronomy II, p.15, indicates a period between 1400 and 900 BCE. Schaefer, 2007, The Latitude and Epoch for the Origin of the Astronomical Lore in MUL.APIN, the period around the year 1370 BCE.

13 The oldest known source from 430 BCE is Herodotus, who cites this in Chapter 109 of book II (2.109.3). See also Euclid, Elements, book II, pp.370–371. For more historical sources, see also Gibbs, 1973, Greek and Roman Sundials, pp. 5–11. The oldest partially preserved Greek sundial is planar and originates from the 4th century BCE, probably from Athens or the surrounding area. It contains an indication of the name Theophilus. Theophilus ruled Athens in 348/347 BCE. Schaldach, 2004, The Arachne of the Amphiareion and the Origin of Gnomonics in Greece, particularly p.247.

14 Suidas (Suda), gr. Σουΐδας, a Byzantine encyclopaedia from the 10th century, which contains a lot of information about the later lost ancient works. The authors of the encyclopaedia are not known, nor is it clear whether Suidas (Suda) was an actual person or just the title of the encyclopaedia.

15 Diogenes Laertius II.1. Favorinus, lat. *Favorinus*, a Roman philosophical sceptic and rhetorician, worked in the 2nd century.

16 Szabó, 1992, Das geozentrische Weltbild, pp. 71–72. Vitruvius, De Architectura, 1.6, p.26. Diogenes otherwise said that Anaximander invented the gnomon, which is not true since it was known and used in Egypt and Mesopotamia, as well as in China and probably in India.

17 Powell, 1990, Masse und Gewichte, Reallexikon der Assyriologie und vorderasiatischen Archäologie, pp.457–468, Powell, 1976, Sumerian area measures and the alleged decimal substratum, pp.165–221, Friberg et al., 1990, Seed and Reeds – A Metro-Mathematical Topic Text from Late Babylonian Uruk, pp.482–557.

18 For the preserved tables, see Thureau-Dangin, 1910, Inventaire des Tablets de Tello conservées au Musée Impér. Ottoman, Paris, pp.11, 1175.

danna corresponds to approximately 11 kilometres or seven miles, with 12 such units required for a day-long walk or for one turn of the celestial arch.[19] The day was divided into twelve double hours, which were also called *danna*. At least from the 7th century BCE onwards, these units described the time elapsed in relation to the time of the sunrise or sunset, which is evident from royal records and letters, as well as from ritual, astrological and other records. These units were then subdivided into 30 smaller units, called UŠ,[20] so that one turn of the celestial arch covered 360 UŠ, where 1 UŠ = 1° or 4 minutes, corresponding to the division of the circle into 360° as we know it today.[21] The measurement system in Mesopotamia, which was initially used for counting and measuring, evolved for centuries, and the beginnings of the sexagesimal system, which are apparently based on the Sumerian measurement system, are still poorly studied. Among the oldest preserved records with base 60 are records of counting people and sheep. About 2500 BCE, the ratio between the units for weight, the *mina* and the *shekel*, was a ratio of 2 × mina = 60 × shekel. Similarly, the measurement system for measuring length also developed over the centuries. The Sumerian measurement system underwent several changes and supplements leading to the sexagesimal system probably around 2050 BCE, which is also useful for the purposes of calculation.[22]

Stars and the zodiac

We conclude from cave paintings that our ancestors were following the stars in the Stone Age, but it was certain that calendar making or the monitoring of the seasons was already used in the era of foraging. For the needs of agriculture, ancient people used simple calendaring and time, useful for agricultural purposes, based on the eastern and western stars or constellations. Similar to the measurement system, agronomy also developed over the centuries, and along with the knowledge and understanding of the ecliptic, conditions were also given for the division of the ecliptic, first into even parts and then into 30° parts. The history of the division of the ecliptic into parts dates back to the 2nd millennium BCE and it is based on the records of the eastern stars. It is probably also related to the division of the day

19 Also the *danna* or Akadic *bēru* ≈ 10.8 km or ≈ 60 Greek stadia. Neugebauer, 1983, Astronomy and History, selected essays, pp.16–17.

20 UŠ means length. Neugebauer, 1938, Untersuchungen zur Geschichte der Antiken Astronomie III, or Neugebauer, 1941, Some Fundamental Concepts in Ancient Astronomy, p.16, reprinted in Neugebauer, 1983, p.8. Sachs, Hunger, 1988, Astronomical Diaries and Related Texts. Vol I, Diaries from 652 B. C. to 262 B. C. Hunger, Pingree, 1999, Astral Sciences in Mesopotamia, pp.44–50.

21 For the development of the sexagesimal system, see Neugebauer, 1929, Zur Entstehung des Sexagesimalsystems and Thureau-Dangin, 1938, Sketch of a history of the sexagesimal system.

22 Robson, 2008, Mathematics in Ancient Iraq, pp. 75–84.

into time units.[23] In Ancient India, the stars were well-known at least 3000 years BCE to the astronomers of the Harappan culture.[24] Over the centuries, knowledge has increased and later led to the division of the ecliptic into even parts. In Indian sources, the division of the ecliptic into even parts is described by the concept of *nakṣatras*.[25] Even in the oldest Vedic sources, the term *nakṣatras* is also used for stars or for 27 (or sometimes 28) groups of stars with which the Moon is in conjunction on a given day.[26] We know almost nothing about the calendaring of the oldest cultures, such as the Indus Valley cultures, but there are preserved Indian hymns from the Vedic period proving that the Indians knew about the division of the year into 12 and 360 parts very early on.[27] From the sources from the *Vedic period*, the division of the day into 30 parts is also derived, which is unique to the Indian culture.[28]

The earliest recorded signs of the zodiac originate from Mesopotamia, and they were recorded in 419 BCE.[29] Unlike the Mesopotamian zodiac, which has evolved for centuries, the Greeks summarized the basics of the zodiac with some changes taken from the Babylonians, some of which are most likely to have originated in Egypt, while some are Greek. The notion of the Greek animal circle is found in *Parapegmas*, the precursors of the modern almanacs, EUCTEMON and EUDOXUS, which determine the seasonal calendar on the basis of the ascensions of the constellations.[30] The first person who we know systematically used the signs of the zodiac in the *Parapegma* was CALLIPPUS, in about 330 BCE. A very good description of the meaning of the signs of the zodiac was presented by GEMINUS in chapter 1 of the book *Introduction to the Phenomena*.[31]

23 Recorded in the table VAT 4924. Van der Waerden, 1952–1953, History of the Zodiac, in Van der Waerden, 1951, Babylonian Astronomy III. From the MUL.APIN tables, it follows that the Babylonians knew the trajectory of the Moon through 17 constellations, which were replaced by 12 constellations by the 5th century BCE. Brack-Bernsen and Hunger, 1999, p.280.

24 Parpola, 2013, Beginnings of Indian Astronomy, with Reference to a Parallel Development in China.

25 Nakṣatras are already mentioned by name in Ṛgveda X.85.13, in Atharvaveda XIX.6.2, and in full in Atharvaveda XIX.7 and in Taittirīya Saṁhitā IV.4.10. Yampolsky, 1950, The Origin of the Twenty-Eight Lunar Mansions. Weinstock, 1949, Lunar Mansions and Early Calendars.

26 *Nakṣatras* refer to *Jyotiṣa Vedāṅga* as parts of the ecliptic. Gondhalekar, 2013, pp.141–143 and 155–162. Mishra, 2005, pp.187–190. Nakṣatras probably originate from the time before the Vedic period.

27 Division of the year into four parts in Ṛgveda I.155.6, division into 360 parts I.164.48, also in Atharvaveda X.8.4. Division into 360 and 720 parts also in Śatapatha Brāhmaṇa X.5.4-4. The emergence of the oldest of the Indian hymns of the Ṛgveda dates back to the 2nd millennium BCE. Ôhashi, 1993, Development of Astronomical Observation in Vedic and Post-Vedic India. From the verses of the Ṛgveda comes the knowledge of the calendar, with the months and seasons, apparently based on observation of the moon. Ṛgveda VII.103.7-9, I.25.8, 10.85.5. For the seasons, see Ṛgveda X.90.6, Śatapatha Brāhmaṇa I.6.1.2-3 and Taittirīya Saṁhitā IV.4. l l. l (*Yajurveda*).

28 1/30 of the day or 48 minutes is a muhūrta. Gondhalekar, 2013, The Time Keepers of the Vedas, pp.92–97. Śatapatha Brāhmaṇa X.4.2.18, for an explanation see Eggeling, 1885, The Śataphata Brāhmana According to the Text of the Mādhyamdina School, Part II. Also Ṛgveda X.189.3. Michel (ed.), 2008, Rig-Veda, das Heilige Wissen Indiens, 2, p.403.

29 For an explanation of the possible origin of the symbols of the signs of the zodiac, see Dewdney, 1931, The Zodiac and Early Astronomy, and Webster, 1940, The Origin of the Signs of the Zodiac: An Interpretation from the Psychological Viewpoint.

30 Euctemon's Parapegma is dated about 430 BCE. Eudoxus' work is said to be based on data from the 11th century BCE, i.e. from the time of Assyrian rule. Schaefer, 2004, The latitude and epoch for the origin of the astronomical lore of Eudoxus.

31 Evans, Berggren, 2006, Geminos's Introduction to Phaenomena, pp.113–136.